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A Direct Solver for a Class of Symmetric Linear Systems

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Abstract—A direct method is proposed for the numerical solution of a wide class of symmetric block-tridiagonal systems of linear equations resulting from a finite difference or finite element discretization of elliptic boundary value problems with Dirichlet boundary value conditions on rectangular domains. The method is based on a result which generalizes the classical Cramer's rule, and on an extensive use of the modified Chebyshev polynomials for the formulation of the algorithm. Numerical examples are given to illustrate the validity of the proposed approach.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Consider the following system of linear equations:

$$-U_{j-1} + AU_j - U_{j+1} = F_j; \qquad 1 \le j \le N, \tag{1}$$
$$U_0 = \alpha; \quad U_{N+1} = \beta; \qquad \alpha, \beta \in \mathbb{R}^N, \tag{2}$$
$$U_i = (U_{1i}, U_{2i}, \dots, U_{Ni})^t; \qquad F_i = (F_{1i}, F_{2i}, \dots, F_{Ni})^t.$$

A is a tridiagonal symmetric matrix,

$$A = [\gamma\, \xi\, \gamma]; \qquad F_i, U_i \in \mathbb{R}^N; \qquad \gamma, \xi \in \mathbb{R}. \tag{3}$$

They occur in a wide variety of simulation problems arising in various areas of engineering practice and which require the numerical solution of boundary value problems for elliptic partial differential equations. Discretization of such models by a finite difference or finite element method produces symmetric block-tridiagonal systems (SBTS) of linear equations. Due to their practical importance, these systems are a subject of permanent interest [1,2]. Two classes of numerical methods, direct and iterative methods have been developed [3–8] and efficient computer packages are now available [9].

The Fourier transform method [10], the cyclic reduction [11,12] and the marching methods [13,14] are the most efficient direct methods. The originality of these methods lies in the decomposition of the original system (1)–(3) to a set of subsystems with tridiagonal structure for which very efficient algorithms exist [15]. For all these methods, the number of elementary operations is generally $O(N^2)$.

In this paper, two main theoretical results are presented, which raise the opportunity to generalize the classical Cramer's rule to SBTS of equation. A direct method is proposed which reduces the initial system to a sequence of diagonal systems in the eigenvector coordinate system. An elementary transformation to the original coordinate system is then carried out. Sample applications are provided.

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Again consider the system (1)–(3). Subsequently, I denotes the unity matrix. Let $J_n(x)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$ be the modified n^{th} order second kind Chebyshev polynomial defined in [13,14,16] as follows:

$$J_n(x) = x\,J_{n-1}(x) - J_{n-2}(x), \tag{4}$$

$$J_0(x) = 1, \qquad J_1(x) = x. \tag{5}$$

It has been shown [14] that

$$J_n(x) = \prod_{k=1}^{n} (x - R_n(k)); \qquad R_n(k) = 2\cos\left(\frac{k\pi}{n+1}\right). \tag{6}$$

We define the operation Δ^* on vectors as follows.

$\Delta^* U_i$ is obtained by applying the rules for computing the determinant associated with an unknown in Cramer's rule. As an example, consider the case $m = 3$:

$$\Delta^* U_i = \begin{vmatrix} F_1 & -I & 0 \\ F_2 & A & -I \\ F_3 & -I & A \end{vmatrix} = \begin{vmatrix} A & -I \\ -I & A \end{vmatrix} F_1 - \begin{vmatrix} -I & 0 \\ -I & A \end{vmatrix} F_2 + \begin{vmatrix} -I & 0 \\ A & -I \end{vmatrix} F_3 = J_2 F_1 + A F_2 + F_3. \tag{7}$$

2. THEORETICAL RESULTS

There are two main results which we state as theorems.

THEOREM 1.

$$\begin{gathered} \Delta^* U_i = J_{N-i}\left(\sum_{k=1}^{i-1} J_{k-1} F_k\right) + J_{i-1}\left(\sum_{k=i}^{N} J_{N-k} F_k\right), \\ J_k \equiv J_k(A), \qquad k = 1, 2, \dots, N. \end{gathered} \tag{8}$$

PROOF. This is easily seen from the properties of the modified Chebyshev polynomials [13,14] and after elementary algebraic manipulations.

REMARK. For the system

$$U_{i-1} + AU_i + U_{i+1} = F_i; \qquad U_0 = \alpha; \quad U_{N+1} = \beta,$$

relation (8) becomes:

$$\Delta^* U_i = J_{N-i}\left(\sum_{k=1}^{i-1} J_{k-1} F_k (-1)^{i+k}\right) + J_{i-1}\left(\sum_{k=i}^{N} J_{N-k} F_k (-1)^{i+k}\right). \tag{9}$$

THEOREM 2. *Consider the system (1), (2). For $N \in \mathbb{N} - \{0,1\}$. It may be solved using the relation*

$$U_i = J_N^{-1} \Delta^* U_i; \qquad i = 1, 2, \dots, N. \tag{10}$$

PROOF. The proof is by induction. Let P(N) be the **proposition** to be verified. Obviously P(2) and P(3) are true. Now suppose that P(N-1) is true. Let us examine P(N). Let (1)–(3) be the system associated with P(N). Suppose that the solution U_1 is already obtained by carrying out the necessary substitutions. Setting

$$V_i = U_{i+1} (i = 1, 2, \dots, N-1); \qquad G_1 = F_2 + U_1; \qquad G_i = F_{i+1} (i = 2, 3, \dots, N-1),$$

system (1) and (2) reduces to:

$$-V_{i-1} + AV_i - V_{i+1} = G_i; \qquad i = 1, 3, \dots, N-1; \qquad V_0 = V_N = 0. \tag{11}$$

Note that the system (11) is similar to (1), but of dimension reduced by one. Thus, (10) may be applied.

$$J_{N-1}\, V_1 = \Delta^* U_i = \sum_{k=1}^{N} J_{n-1-k}\, G_k, \tag{12}$$

therefore,

$$\left(A J_{N-1} - J_{N-2}\right) U_2 = -J_{N-2} F_1 + A \sum_{k=2}^{N} J_{N-k} F_k \tag{13}$$

and

$$J_N U_2 = \left(\sum_{k=1}^{N-1} J_{k-1} F_k \right) J_{N-2} + J_1 \left(\sum_{k=2}^{N} J_{N-k} F_k \right). \tag{14}$$

This shows that $P(N)$ is satisfied by U_2. Let us do the same with U_1. (1) and (2), after algebraic manupulations and taking into account (10), yield:

$$J_N U_1 = J_{N-1} F_1 - \sum_{k=2}^{N} J_{N-k} F_k = \sum_{k=1}^{N} J_{N-k} F_k. \tag{15}$$

This shows that $P(N)$ is also satisfied by U_1. Now, suppose that $P(N)$ is satisfied by U_{i-2} and U_{i-1}. From the previous results and by analogy with the case of U_2, the result (10) is easily obtained. This concludes the proof of theorem 1. Let us at this point worry about computer implementation of the algorithm.

3. SOLUTION ALGORITHM

The eigenvalues of A are [16]

$$\lambda_j = \xi + 2\gamma \cos \frac{j\pi}{N+1}; \qquad j = 1, 2, \ldots, N, \tag{16}$$

and the corresponding eigenvector matrix is

$$P = (P_{ij}) = \left(\left(\frac{2}{N+1} \right)^{1/2} \sin \left(\frac{ij\pi}{N+1} \right) \right). \tag{17}$$

Note that P is orthogonal and J_m and A have the same eigenvector coordinate system. Also, we have

$$P^t A P = \Lambda = \mathrm{Diag}(\lambda_i). \tag{18}$$

Let us describe some steps necessary for the computer implementation of the method.

(i) Determination of the Eigenvalues of J_N

Let β_{kL} be the k^{th} eigenvalue of J_L. From (4), it follows that

$$\beta_{kN} = \lambda_k\, \beta_{kN-1} - \beta_{kN-2}. \tag{19}$$

Equation (19) relates the eigenvalues β_{kN} of J_N to second kind Chebyshev polynomial in l. Thus, applying (6) yields

$$\beta_{kN} = \prod_{j=1}^{N} \left(\lambda_k - R_N(j) \right); \qquad k = 1, 2, \ldots, N. \tag{20}$$

The computation of β_{kN} using (20) supposes that the factors have been previously ordered. Ordering algorithms are proposed in [17].

(ii) Determination of U_1 in the Eigenvector Coordinate System

From (10) and (8) follows, in the eigenvector coordinate system:

$$U_{i1} = \frac{1}{\beta_{iN}} \left[\sum_{k=1}^{N} \beta_{iN-k} \mathbb{F}_i k \right]; \qquad i = 1, 2, \dots, N, \tag{21}$$
$$\mathbb{F}_i = (\mathbb{F}_{1i}, \mathbb{F}_{2i}, \dots, \mathbb{F}_{Ni})^t = P^t F_i.$$

(iii) Determination of $U_j (2 \le j \le N)$ in the Eigenvector Coordinate System

Equation (1) yields, in the eigenvector coordinate system:

$$U_{i2} = F_{i1} - \lambda_i U_{i1}; \qquad i = 1, 2, \dots, N, \tag{22}$$
$$U_{ij} = Fij - 1 - (U_{ij-2} + \lambda_i U_{ij-1}); \qquad j > 2; \qquad i = 1, 2, \dots, N-1. \tag{23}$$

This concludes the solution of the equation in the eigenvector coordinate system.

(iv) Determination of the Transformation of the Solution from the Eigenvectors Coordinate System to the Original Coordinate System

Let $X = (X_1, X_2, \dots, X_N)$ and $Y = (Y_1, Y_2, \dots, Y_N)$ be, respectively, the solution in the eigenvector coordinate system and in the original coordinate system. Thus,

$$Y_i = P X_i; \qquad i = 1, 2, \dots, N.$$

The total number of elementary operation is $O(N^3)$.

4. SAMPLE APPLICATIONS

The aim of the proposed examples is simply to demonstrate the validity of the results (8)–(10) presented in Theorems 1 and 2.

EXAMPLE 1. Consider the system of linear equations: $A\ U = F$, $A = [-1\ 4\ \ -1]$, $F = [30\ 20\ 20\ 30]^t$; $U = (U_1, U_2, U_3, U_4)$. The exact solution is $U_1 = U_2 = U_3 = U_4 = 10$. Equations (4), (5), and (8) yield:

$$J_0 = 1, \quad J_1 = 4, \quad J_2 = 15, \quad J_3 = 56, \quad J_4 = 209;$$
$$\Delta^* U_1 = J_3 F_1 + J_2 F_2 + J_1 F_3 + F_4 = 2090,$$
$$\Delta^* U_2 = J_2 F_1 + J_1 J_2 F_2 + J_1 J_1 F_3 + J_1 F_4 = 2090,$$
$$\Delta^* U_3 = J_1 \{J_0 F_1 + J_2 F_2\} + J_2 \{J_1 F_3 + F_2\} = 2090,$$
$$\Delta^* U_4 = J_0 F_1 + J_1 F_2 + J_2 F_3 + J_3 F_4 = 2090.$$

Therefore, (10) yields $U_1 = U_2 = U_3 = U_4 = 10$.

EXAMPLE 2. Consider the block-tridiagonal system:

$$\begin{cases} \mathbf{A} u_1 - \mathbf{I} u_2 & = F_1 \\ -\mathbf{I} u_1 + \mathbf{A} u_2 - \mathbf{I} u_3 & = F_2 \, , \\ -\mathbf{I} u_2 + \mathbf{A} u_3 & = F_3 \end{cases}$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix},$$
$$F_1 = F_3 = (20, 10, 20)^\top, \quad F_2 = (10, 0, 10)^\top.$$

The exact solution is: $U_1 = U_2 = U_3 = (10, 10, 10)^\top$. Applying the proposed method yields:

$$J_0 = I, \quad J_1 = A, \quad J_2 = \begin{pmatrix} 16 & -8 & 1 \\ -8 & 17 & -8 \\ 1 & -8 & 16 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 68 & -48 & 12 \\ -48 & 80 & -48 \\ 12 & -48 & 68 \end{pmatrix},$$

$$\begin{aligned} \Delta^* U_1 &= J_2 F_1 + J_1 F_2 + F_3 = (320, -160, 320)^\top; \\ \Delta^* U_2 &= J_1 F_1 + J_1 J_1 F_2 + J_1 F_3 = (320, -160, 320)^\top; \\ \Delta^* U_3 &= F_1 + J_1 F_2 + J_2 F_3 = (320, -160, 320)^\top; \\ J_3^{-1} &= \begin{pmatrix} 0.03125 & 0.02678 & 0.01339 \\ 0.02678 & 0.04462 & 0.02678 \\ 0.01339 & 0.02678 & 0.03125 \end{pmatrix}. \end{aligned}$$

Therefore, (10) yields: $U_1 = U_2 = U_3 = (10, 10, 10)^\top$. Now we apply algorithm (16)–(23).

The eigenvalues of A are

$$\lambda_1 = 2.5857; \qquad \lambda_2 = 4; \qquad \lambda_3 = 5.41421.$$

The matrix of eigenvectors is

$$p = \begin{pmatrix} 0.50 & 0.707106 & 0.5 \\ 0.707106 & 0 & -0.707106 \\ 0.5 & -0.707106 & 0.5 \end{pmatrix}.$$

The eigenvalues of the J_i are

$$\begin{aligned} &\beta_{11} = 2.5857 && \beta_{12} = 5.6862 && \beta_{13} = 121.1774, \\ &\beta_{21} = 4 && \beta_{22} = 15 && \beta_{23} = 56, \\ &\beta_{31} = 5.4142 && \beta_{32} = 28.3137 && \beta_{33} = 147.8822. \end{aligned}$$

The vectors $\mathbb{F}_k$ are

$$\mathbb{F}_1 = \mathbb{F}_3 = (2.7071, 0, 1.2928)^\top; \quad \mathbb{F}_2 = (1, 0, 1)^\top.$$

The solution in the eigenvector coordinate system is: $\mathbb{U}_i = P^t U_i$,

$$\mathbb{U}_1 = \mathbb{U}_2 = \mathbb{U}_3 = (17.07106, 0, 2.928932)^\top.$$

Solution in the original coordinate system $= P\mathbb{U}_i$,

$$U_1 = U_2 = U_3 = (10, 10, 10)^\top.$$

5. CONCLUSION

The paper proposes a simple and original generalization of the Cramer's rule to a more general class of linear symmetric systems of equation. The main theoretical results are given in Theorems (1) and (2). A strategy for computer implementation of the method which reduces the initial system to a sequence of diagonal systems in the eigenvector coordinate system is proposed. The examples presented are symbolic but fully illustrate the validity of the main theoretical results of the paper.

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